Total number of printed pages–7

1 (Sem–4) MAT 4

2025

MATHEMATICS

Paper : MAT0400404

(Number Theory–I)

Full Marks : 60

Time : 2½ hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×8=8

(a) State Well-Ordering Principle.

(b) If a and b are integers with $b \neq 0$, then there exist unique integers q and r such that $a = qb + r$ where

(i) $0 < r \leq b$

(ii) $0 \leq r < |b|$

(iii) $0 \leq r \leq b$

(iv) $0 \leq r \leq |b|$

(Choose the correct option)

(c) Which of the following Diophantine equation cannot be solved ?

(i) $6x + 51y = 22$

(ii) $24x + 138y = 18$

3

(iii) $158x - 57y = 7$

(iv) $221x + 35y = 11$

(d) Give an example to show that $a^2 \equiv b^2 \pmod{n}$ need not imply that $a \equiv b \pmod{n}$.

(e) Without performing the division, determine whether the integer 176, 521, 221, is divisible by 9.

(f) If p is a prime number, then

(i) $(p-1)! \equiv 1 \pmod{p}$

(ii) $(p-1)! \equiv -1 \pmod{p}$

(iii) $(p+1)! \equiv 1 \pmod{p}$

(iv) $(p+1)! \equiv -1 \pmod{p}$

(Choose the correct option)

(g) Find $\sigma(180)$.

(h) Define Möbius μ–function.

2. Answer the following questions : $2\times6=12$

(a) If $a|c$ and $b|c$ with $gcd\ (a,b)=1$, then prove that $ab|c$.

(b) Prove that $gcd\ (a+b, a-b)=1$ or 2 if $gcd\ (a,b)=1$.

(c) Use Fermat's theorem to show that $5^{38} \equiv 4 \pmod{11}$.

(d) Show that 41 divides $2^{20}-1$.

(e) If n is a square free integer, prove that $\tau(n) = 2^r$, where r is the number of prime divisors of n.

(f) For $n>2$, prove that $\phi(n)$ is an even integer.

$2^2\ 3^2\ 5$

3. Answer ***any four*** of the following questions : 5×4=20

(a) State and prove Archimedean property. 1+4=5

(b) Use the Euclidean Algorithm to obtain integers x and y satisfying

$$gcd\,(12378,\ 3054) = 12378x + 3054y$$

(c) Use Chinese Remainder Theorem to solve the simultaneous congruences

$$x \equiv 2(mod\,3)$$

$$x \equiv 3(mod\,5)$$

$$x \equiv 2(mod\,7)$$

(d) If n and r are positive integers with $1 \le r < n$, then prove that the bionomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is also an integer.

(e) Prove that every positive integer $n>1$ can be expressed uniquely as a product of primes a part from the order in which the factors occur.

$$\begin{array}{r} 3054 \\ \times\ 4 \\ \hline 12216 \end{array} \qquad \begin{array}{r} 3054 \\ \times\ 3 \\ \hline 9162 \end{array}$$

(f) If p and q are distinct primes, prove that $p^{q-1}+q^{p-1}\equiv 1 \pmod{pq}$

(g) If f is a multiplicative function and F be defined by $F(n)=\sum_{d|n} f(d)$, then prove that F is also multiplicative.

(h) If $n \geq 1$ and $gcd\ (a,n)=1$, then prove that $a^{\phi(n)} \equiv 1 \pmod{n}$.

4. Answer **any two** of the following questions : 10×2=20

(a) (i) Prove that for given integers a and b, with $b>0$, there exist unique integers q and r satisfying

$a = bq + r,\ 0 \leq r < b$ 6

12378
−12216
162

(ii) Establish the following formula by Mathematical induction.

$$1{\cdot}2+2{\cdot}3+3{\cdot}4+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$

for all $n \geq 1$ 4

35 × 3 = 165

528 + 7 = 535

24 × 5 = 120

24 × 6 = 144

18 × 7 = 126

4 ~~149~~

(b) (i) Given integers a and b, not both of which are zero, prove that there exist integers x and y such that $gcd\ (a, b) = ax + by$. 5

(ii) Determine all solutions in the positive integers of the Diophantine equation

$172x + 20y = 1000$ 5

(c) (i) Prove that if p is a prime and $p \nmid a$, then

$a^{p-1} \equiv 1 (mod\, p)$

Is the converse of it true ? Justify.

5+1=6

(ii) Solve : $9x \equiv 21 (mod\, 30)$. 4

(d) (i) Prove that there is an infinite number of primes. 4

(ii) Prove that the quadratic congruence $x^2 + 1 \equiv 0 (mod\, p)$ where p is an odd prime has a solution if and only if

$p \equiv 1 (mod\, 4)$. 6

(e) (i) If $n = p_1^{k_1} p_2^{k_2} \ldots . p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that

(I) $\tau(n) = (k_1 + 1)(k_2 + 1) \ldots \ldots . (k_r + 1)$

(II) $\sigma(n) = \left(\dfrac{p_1^{k_1+1} - 1}{p_1 - 1}\right)\left(\dfrac{p_2^{k_2+1} - 1}{p_2 - 1}\right) \ldots \ldots \left(\dfrac{p_r^{k_r+1} - 1}{p_r - 1}\right)$

6

(ii) For each positive integer $n \geq 1$, prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

4

1720 → (1)

even no. 160

$1 + 1 = 2$. 12

m is odd no

$(-1)^n = 1 - 1 \epsilon = 0.25$

$|x_n - x_m| = |2 - 0|$

$= 2 > \epsilon$.

3
25
17
175
$25\times$
425